Unit Quaternion 0x05

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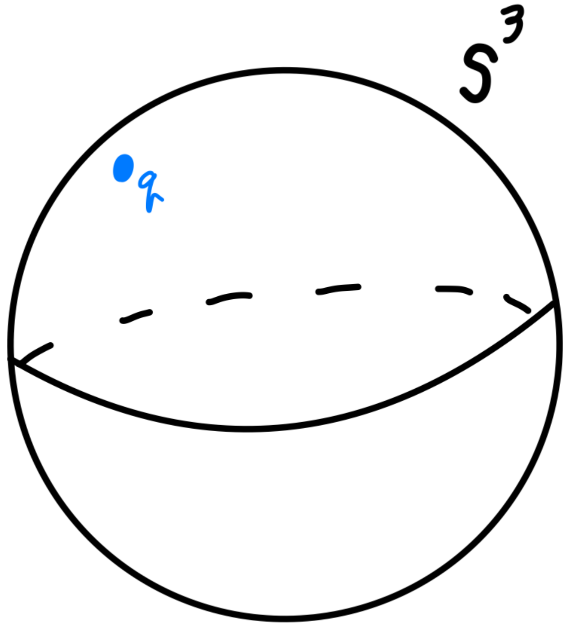
Unit Quaternion

Unit Quaternion

- It represents 3D rotations

$$
\begin{aligned}
q &= w + ix + jy + kz \\
&= (w, x, y, z) \\
&= (w, \vec{v})
\end{aligned}
$$



$$||q||^2 = w^2 + x^2 + y^2 + z^2 = 1$$

Unit Quaternion Algebra

Identity

$$q = (1, 0, 0, 0)$$

Product

$$(w_1, \vec{v_1}) \cdot (w_2, \vec{v_2}) = (w_1 w_2 - \vec{v_1} \cdot \vec{v_2}, w_1 \vec{v_2} + w_2 \vec{v_1} + \vec{v_1} \times \vec{v_2})$$

inverse

$$q^{-1} = (w, -x, -y, -z)$$

$$q^{-1} q \rightarrow q_I$$

Euler parameters

Unit quaternion

$$||q||^2 = w^2 + x^2 + y^2 + z^2 = 1$$

$$q = (e_0, e_1, e_2, e_3)$$

$$e_0 = \cos\left(\frac{\theta}{2}\right)$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \hat{v}\sin\left(\frac{\theta}{2}\right)$$

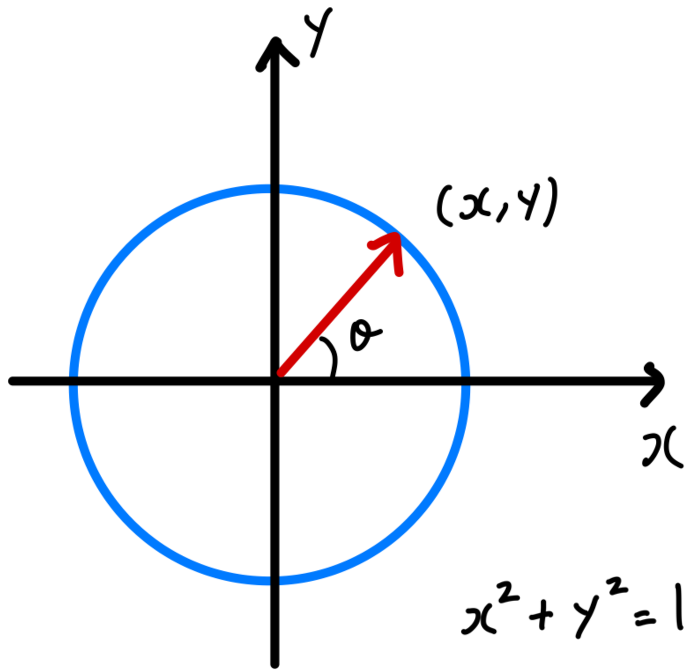
θ: rotation angle

$\hat{v}$: rotation axis

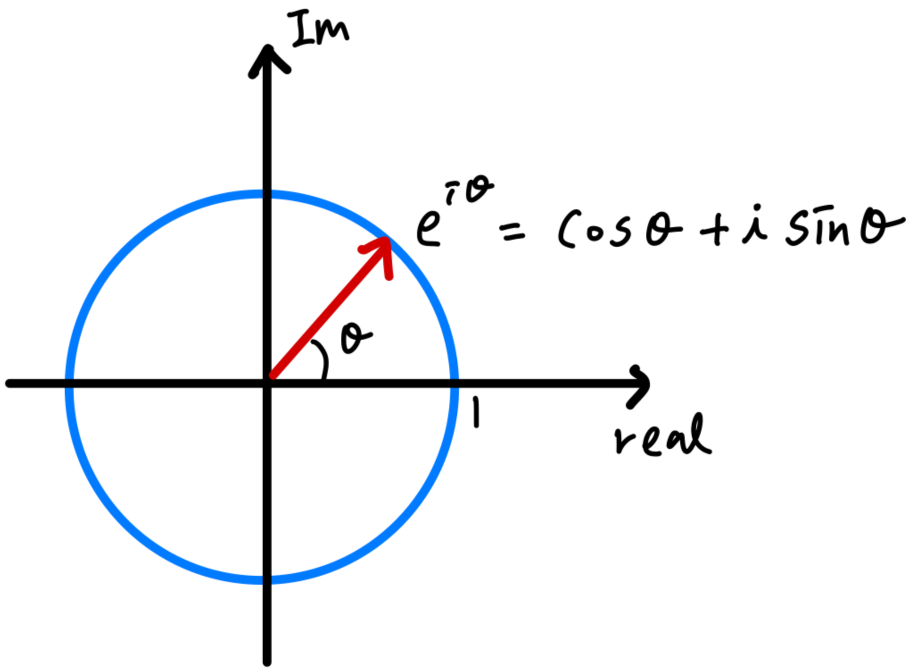
$$q_1 = (\overset{\text{Scalar}}{\boxed{a,}}\underset{\text{Vector}}{\boxed{b, c, d}})$$

Orientation 2D

Complex Plane



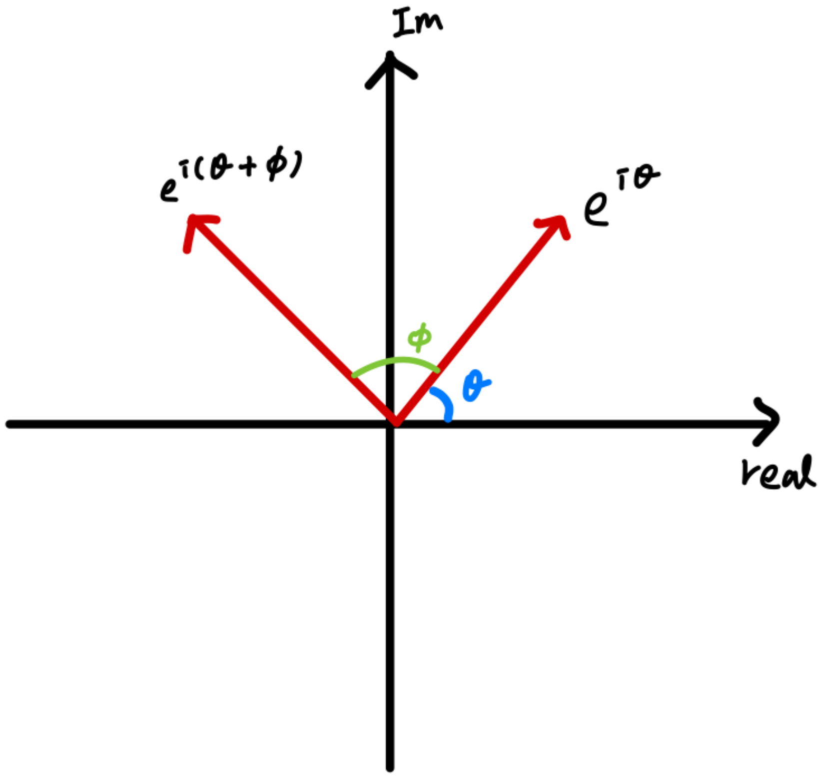
<Cartesian Coordinate>



<Complex Plane>

Orientation 2D

Euler Formula



$$e^{i(\theta+\emptyset)} = e^{i\theta}e^{i\emptyset}$$

Euler parameters

Unit quaternion

$$q = (e_0, e_1, e_2, e_3) \longrightarrow \left(\cos\left(\frac{\theta}{2}\right), \hat{v}\sin\left(\frac{\theta}{2}\right)\right)$$

$$\downarrow$$

$$\cos\left(\frac{\theta}{2}\right) + \hat{v}\sin\left(\frac{\theta}{2}\right) = e^{\hat{v}\frac{\theta}{2}}$$

$$e_0 = \cos\left(\frac{\theta}{2}\right)$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \hat{v}\sin\left(\frac{\theta}{2}\right)$$

θ: rotation angle

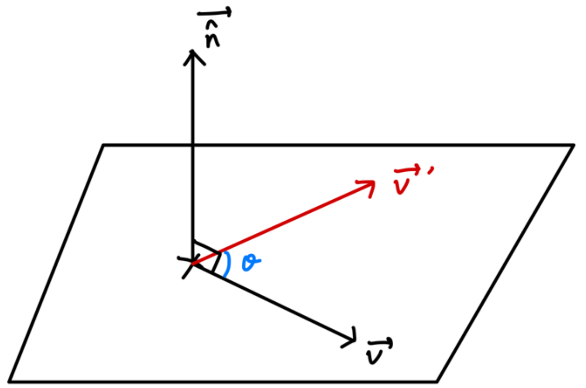
$\hat{v}$: rotation axis

$$\frac{\theta}{2}$$ Why?

Rodrigues Formula

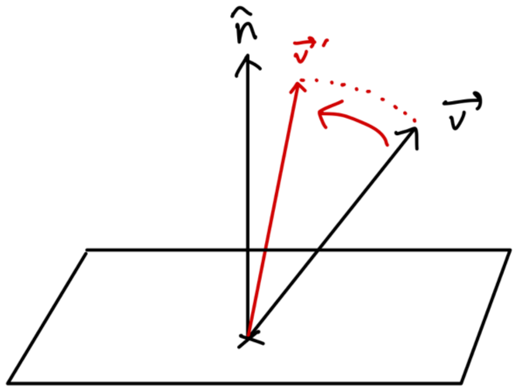
3D rotation

*Special Case



$$\vec{v}' = \cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v})$$

*General Case



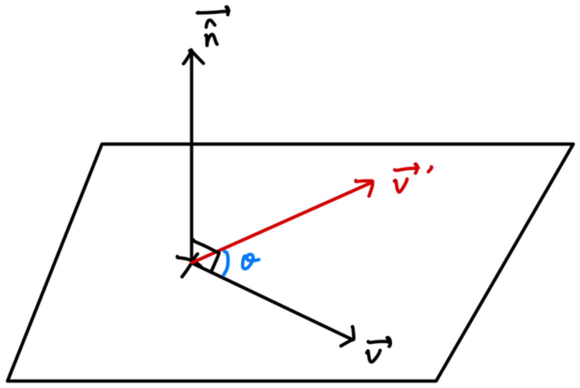
$$\vec{v}' = (1-\cos\theta)(\vec{v}\cdot\hat{n})\hat{n} + \cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v})$$

Quaternion and Rodrigues: Special Case

***Form**

$$v = (0, \vec{v})$$
$$v' = (0, \vec{v'})$$
$$n = (0, \hat{n})$$

$$nv = (-\hat{n} \cdot \vec{v}, \hat{n} \times \vec{v}) = (0, \hat{n} \times \vec{v})$$
$$nv = \hat{n} \times \vec{v}$$



***Vector to Quaternion**

$$v' = \cos\theta \cdot v + \sin\theta \cdot (nv)$$
$$v' = (\cos\theta + \sin\theta \cdot n)v$$
$$v' = e^{\theta n} v$$

$$\vec{v'} = \cos\theta \cdot \vec{v} + \sin\theta \cdot \underline{(\hat{n} \times \vec{v})}$$

$$\boxed{\cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v}) \longrightarrow e^{\theta n} v}$$

Vector → Quaternion

*Special Case

$$\cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v}) \longrightarrow e^{\theta n} v$$

*e.g

*input

$$\vec{v} = (0,0,1)$$
$$\hat{n} = (0,1,0)$$
$$\theta = \frac{\pi}{2}$$

*input

$$v = k$$
$$n = j$$

*output

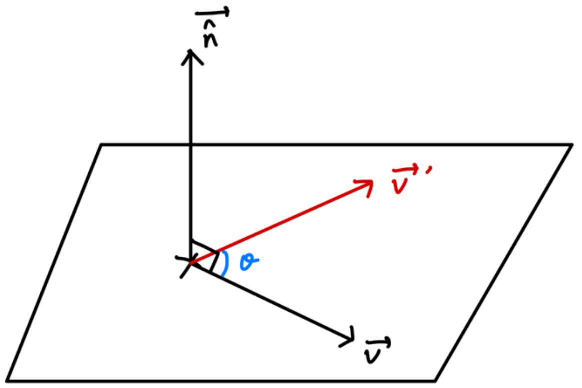
$$\vec{v'} = (?,?,?)$$

*output

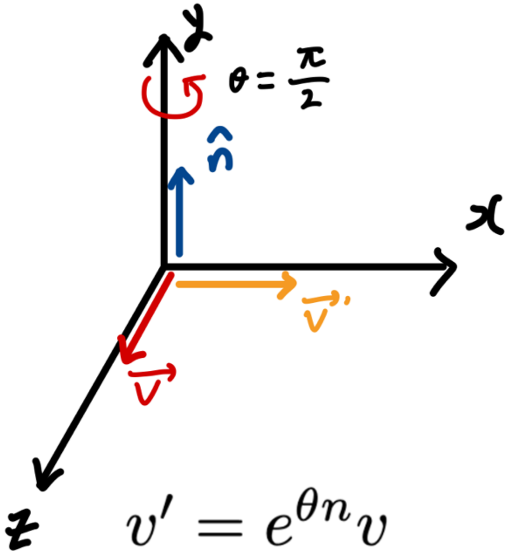
$$v' = ?$$

*Equation

$$v' = e^{\theta n} v \qquad v' = e^{\frac{\pi}{2} j} k \longrightarrow v' = jk \qquad v' = i \longrightarrow v' = (1,0,0)$$



$$\vec{v'} = \cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v})$$



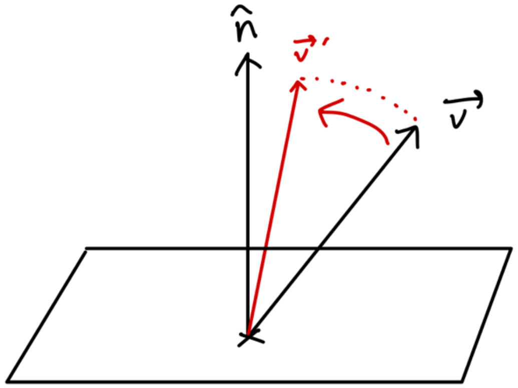
$$v' = e^{\theta n} v$$

Quaternion and Rodrigues: General Case

***Form**

$$v = (0, \vec{v})$$
$$v' = (0, \vec{v'})$$
$$n = (0, \hat{n})$$

$$v_{\parallel} = (0, \vec{v_{\parallel}})$$
$$v_{\perp} = (0, \vec{v_{\perp}})$$
$$v'_{\perp} = (0, \vec{v'_{\perp}})$$



$$\vec{v'} = (1 - \cos\theta)(\vec{v} \cdot \hat{n})\hat{n} + \cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v})$$

$$v' = e^{\frac{\theta}{2}n} v e^{-\frac{\theta}{2}n}$$

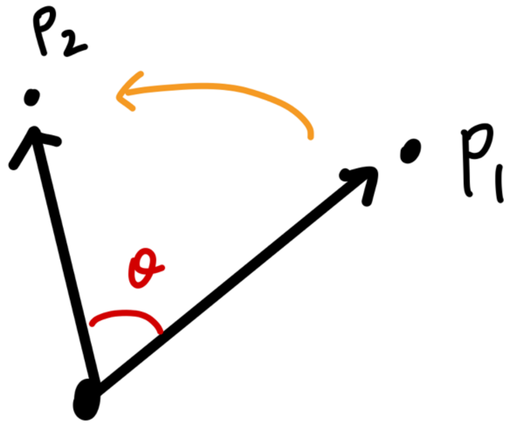
$$\boxed{\cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v}) \longrightarrow v' = e^{\frac{\theta}{2}n} v e^{-\frac{\theta}{2}n}}$$

Rotation using Unit quaternion

***Vector to Quaternion**

$$p_1 = (0, x, y, z)$$

$$q = \left(\cos\left(\frac{\theta}{2}\right), \hat{v}\sin\left(\frac{\theta}{2}\right)\right)$$



$$p_2 = q p_1 q^{-1}$$

$$\boxed{\cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v}) \longrightarrow v' = e^{\frac{\theta}{2}n} v e^{-\frac{\theta}{2}n}}$$

Rotation using Unit quaternion

Rotating p by q_1 and q_2

$$p_1 = q_1 p q_1^{-1}$$

$$p_2 = q_2 p_1 q_2^{-1}$$

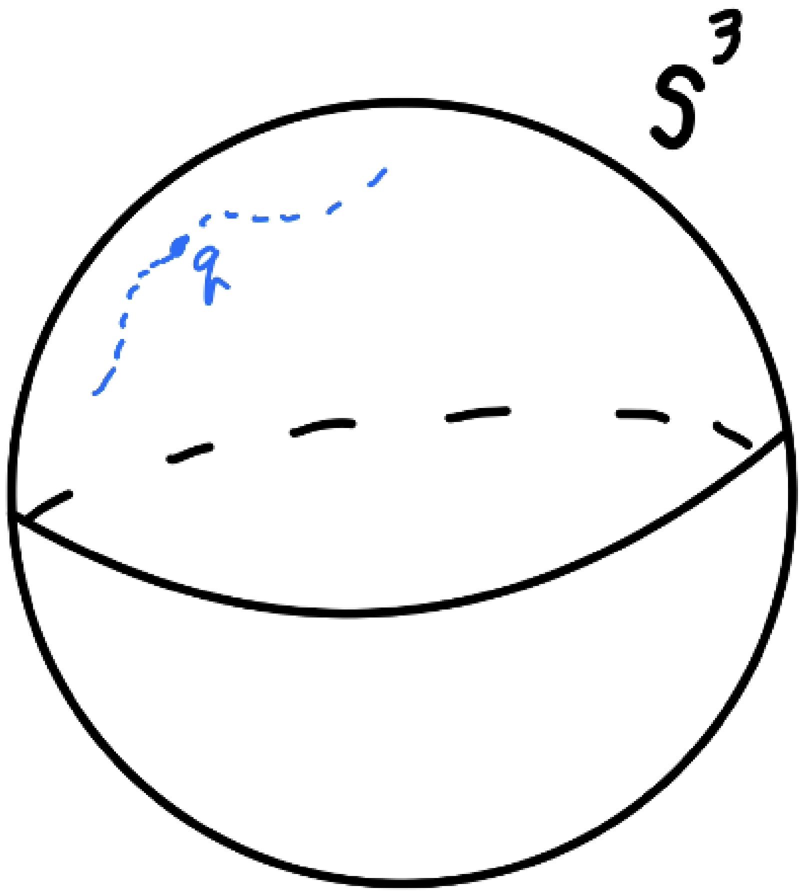
$$q_2 = q_2(q_1 p q_1^{-1})q_2^{-1} = (q_2 q_1)p(q_2 q_1)^{-1}$$

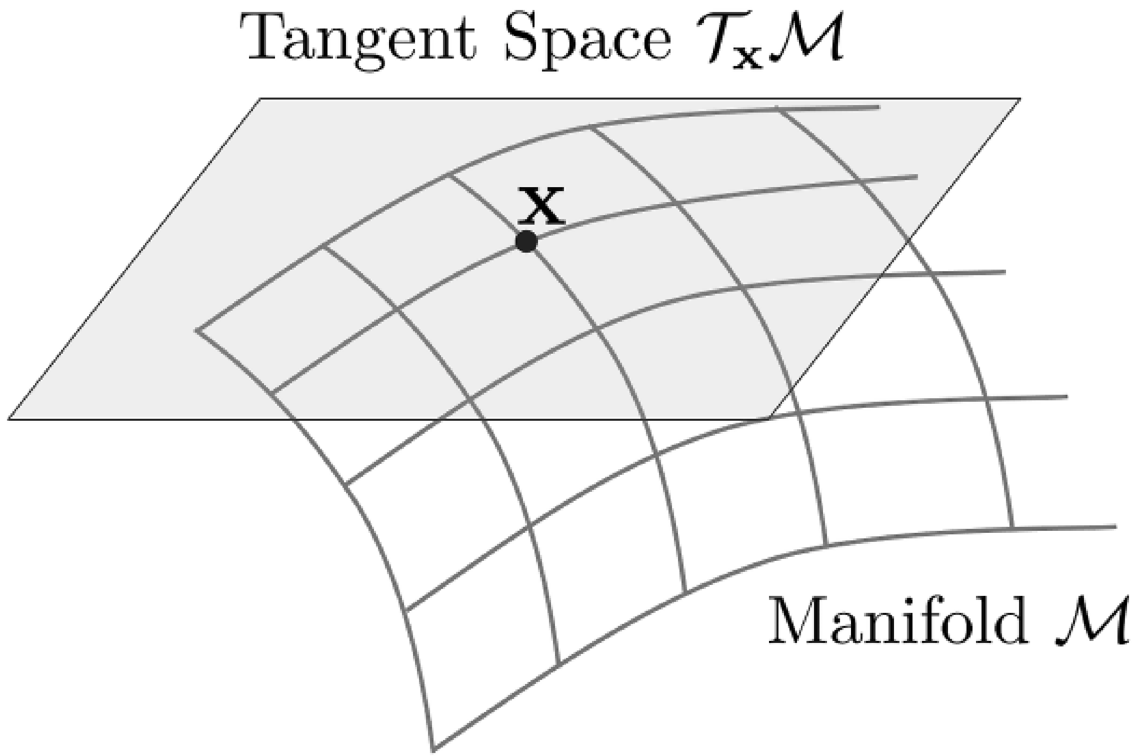
summary

$$q = q_2 q_1$$

$$\mathrm{p}_2 = qpq^{-1}$$

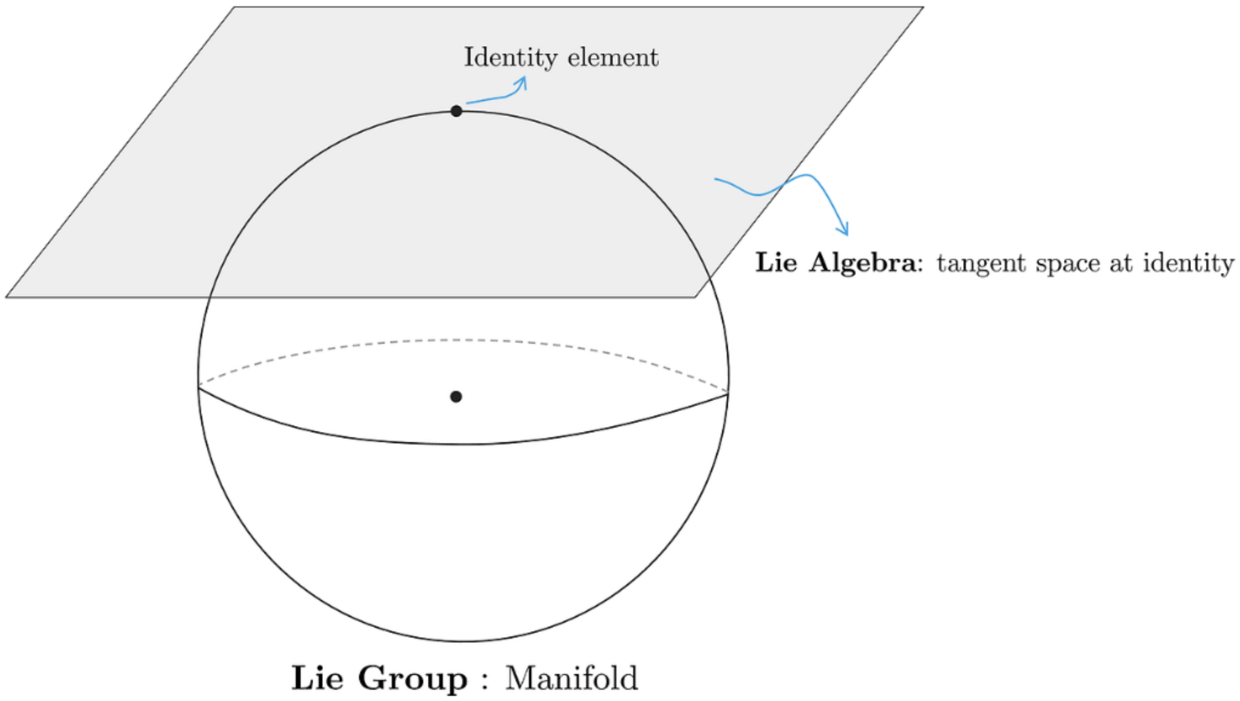
Tangent space of S^3





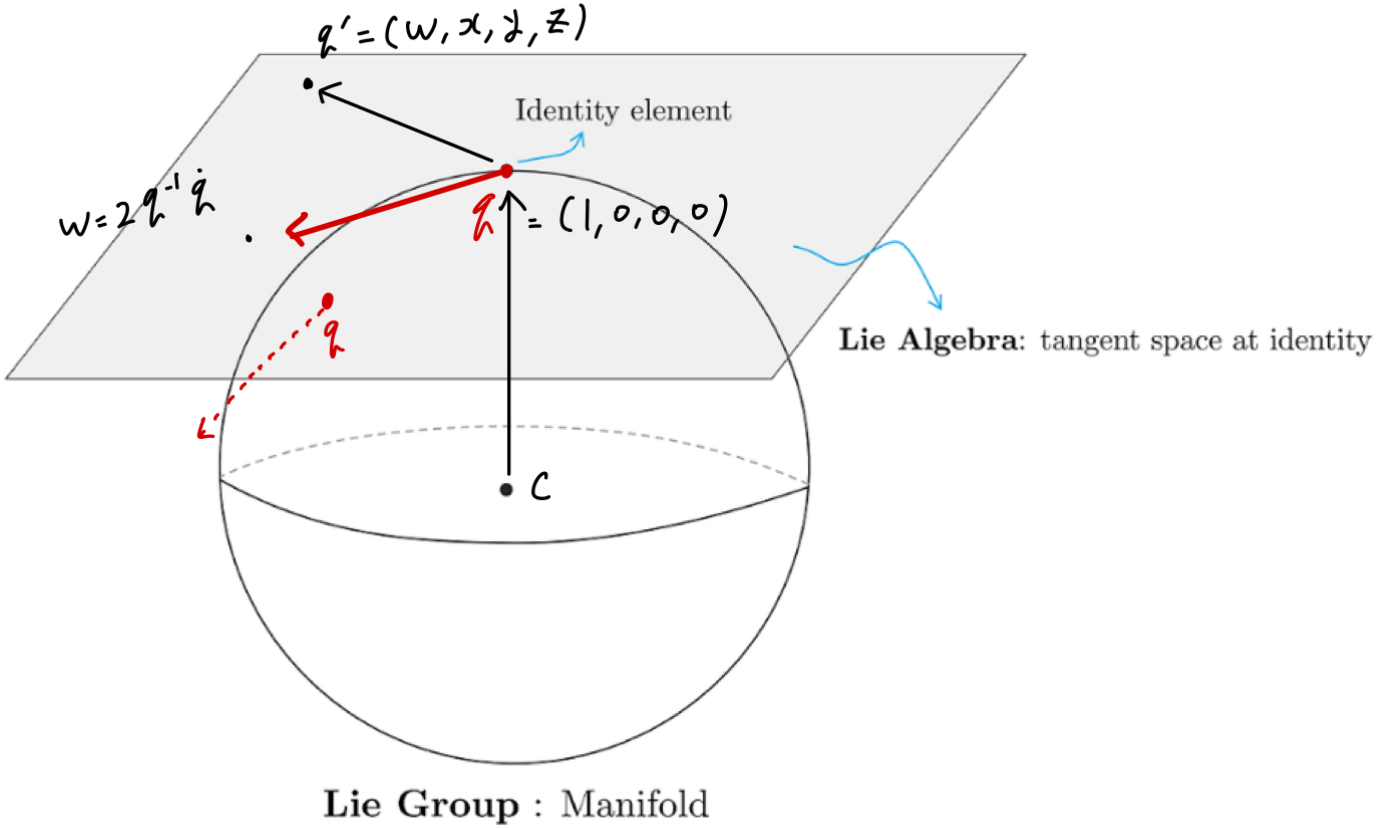
Tangent space of S^3

- The tangent space is uniquely determined for any point
- The dimension of the tangent space is determined DoF of the manifold



The tangent space of Identity element is specifically called Lie Algebra

Tangent space of S^3



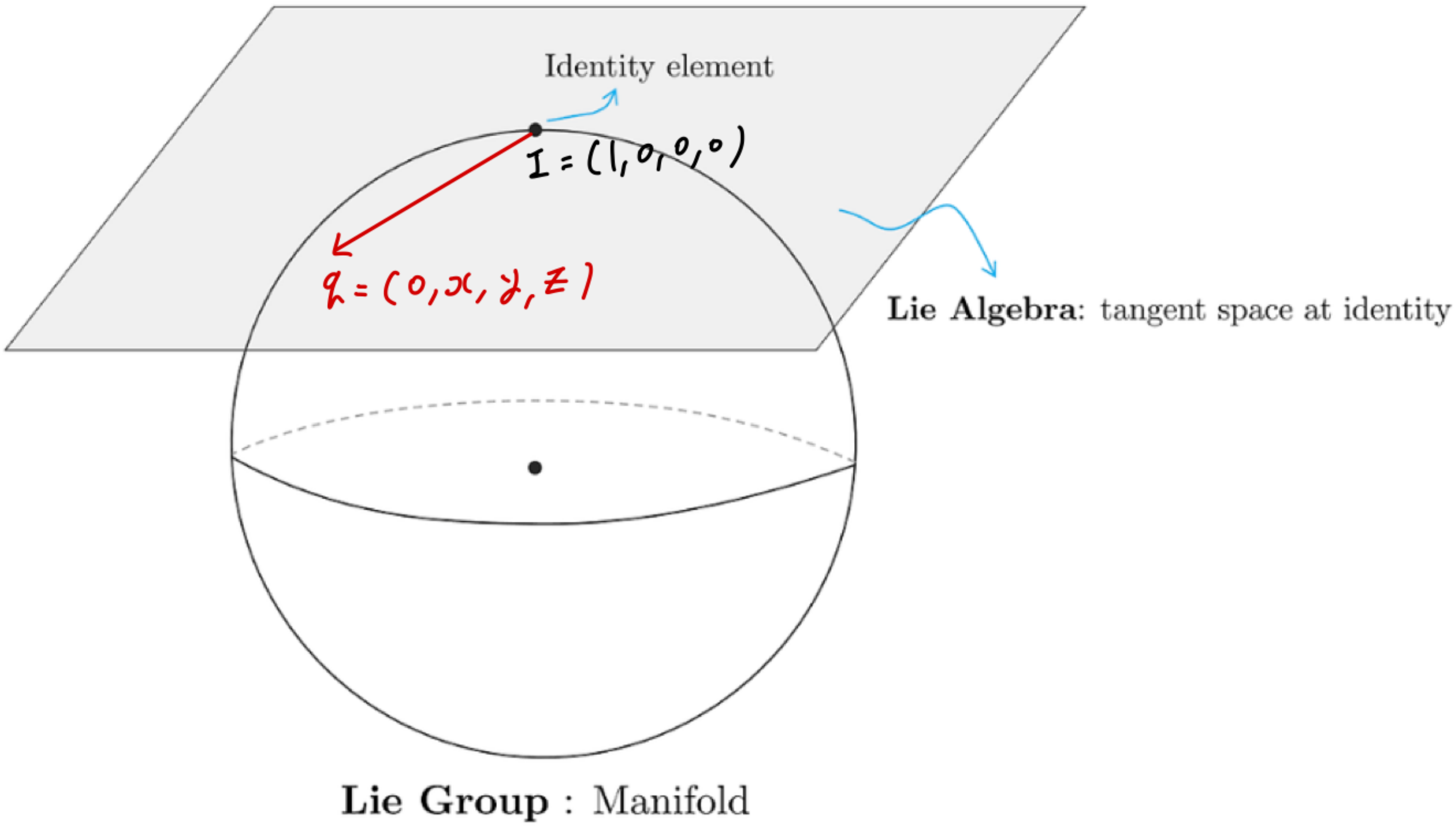
$$q' \cdot q = (w, x, y, z) \cdot (1, 0, 0, 0) = 0$$

$$w = 0$$

Lie Algebra is 3-vector space

Axis-angle(rotation vector)

Tangent space of S^3

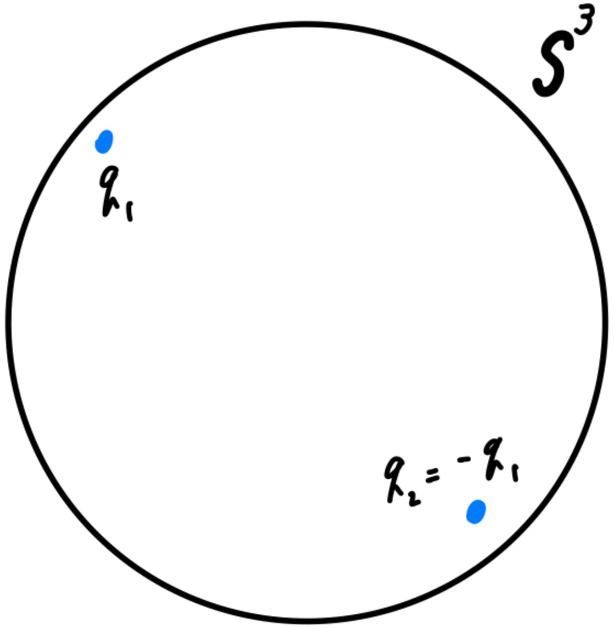


Antipodal Equivalence

The two quaternions represent the same orientation

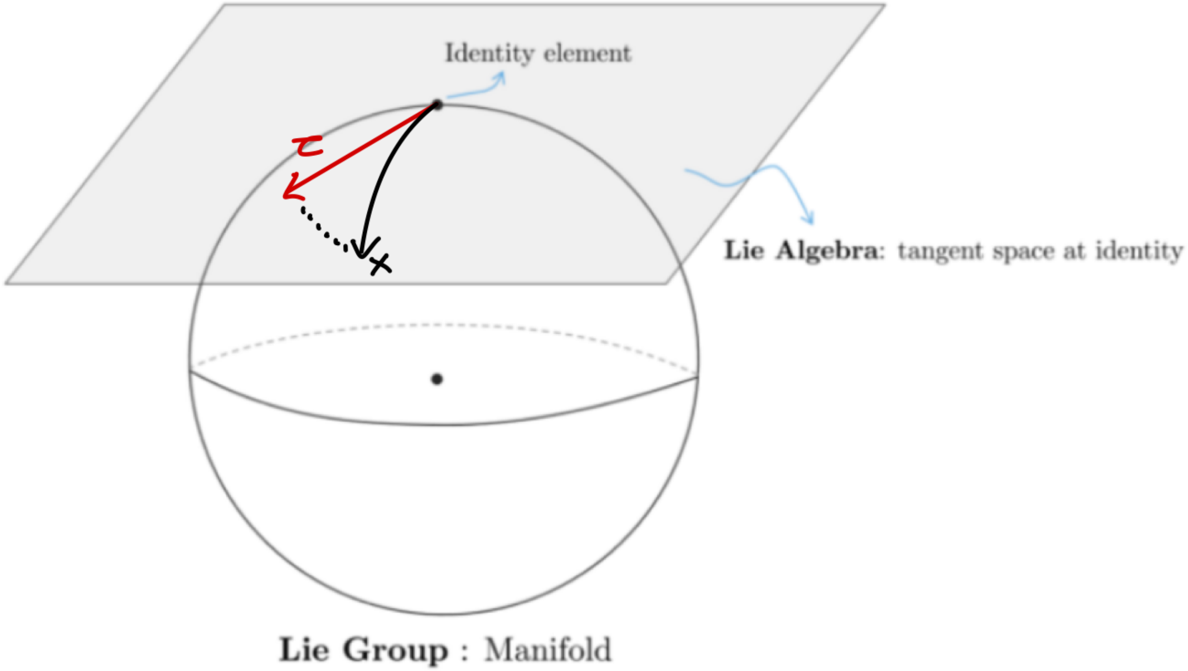
$$q_2 = -q_1$$

$$q_2 p q_2^{-1} = q_1 p q_1^{-1}$$



- The double cover of the manifold of SO(3)
- 2-to-1 mapping

Exponential & Logarithmic Map



Lie Group : 까다로운 제약 조건
Lie Algebra: 비교적 자유로운 제약 조건

Lie Group ←→ Lie Algebra

연산 과정

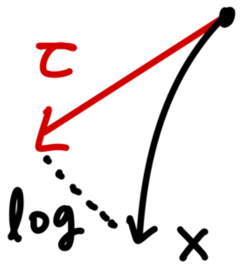
Lie Group → Lie Algebra → Lie Group

↓

Operation

Exponential & Logarithmic Map

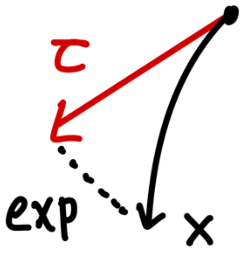
Logarithmic Mapping



Lie Group → Lie Algebra

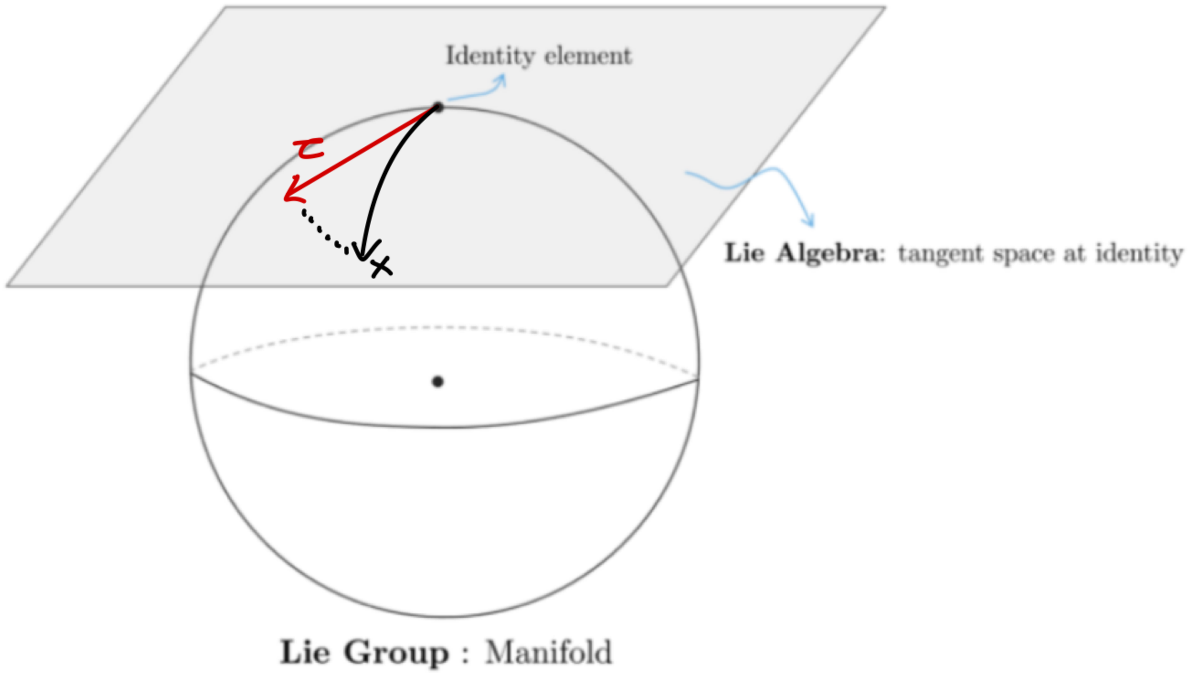
$X \longrightarrow \tau$

Exponential Mapping



Lie Algebra → Lie Group

$\tau \longrightarrow X$



Exponential & Logarithmic Map

exp & Log Mapping at axis-angle

$$e^{\hat{n}\theta} = \cos\theta + \hat{n}\sin\theta$$

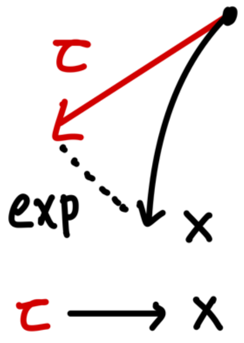
$$log(e^{\hat{n}\theta}) = \hat{n}\theta$$

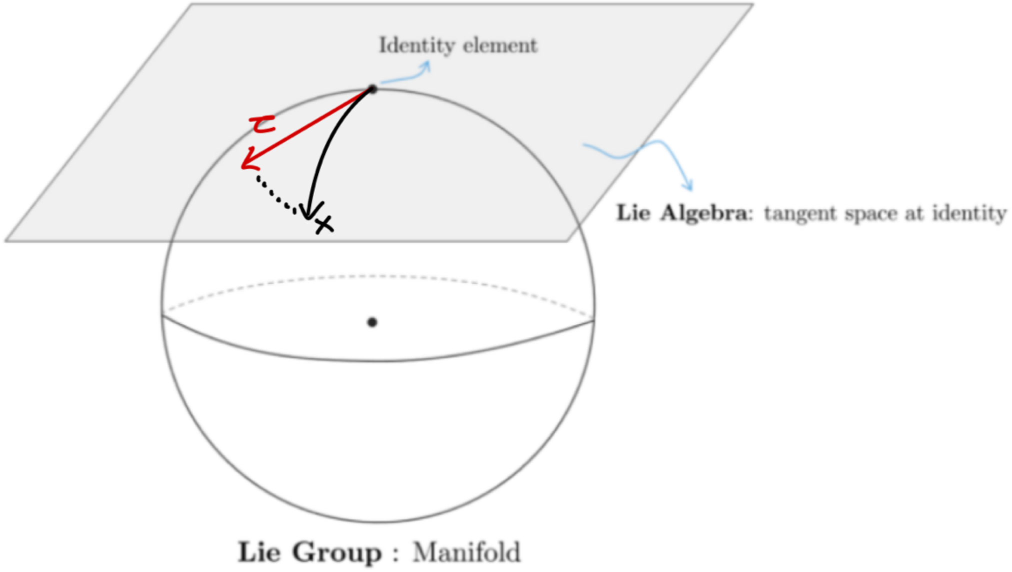
exp & Log Mapping at unit quaternion

$$q = (e_0, e_1, e_2, e_3) \longrightarrow \left(\cos\left(\frac{\theta}{2}\right), \hat{v}\sin\left(\frac{\theta}{2}\right)\right)$$

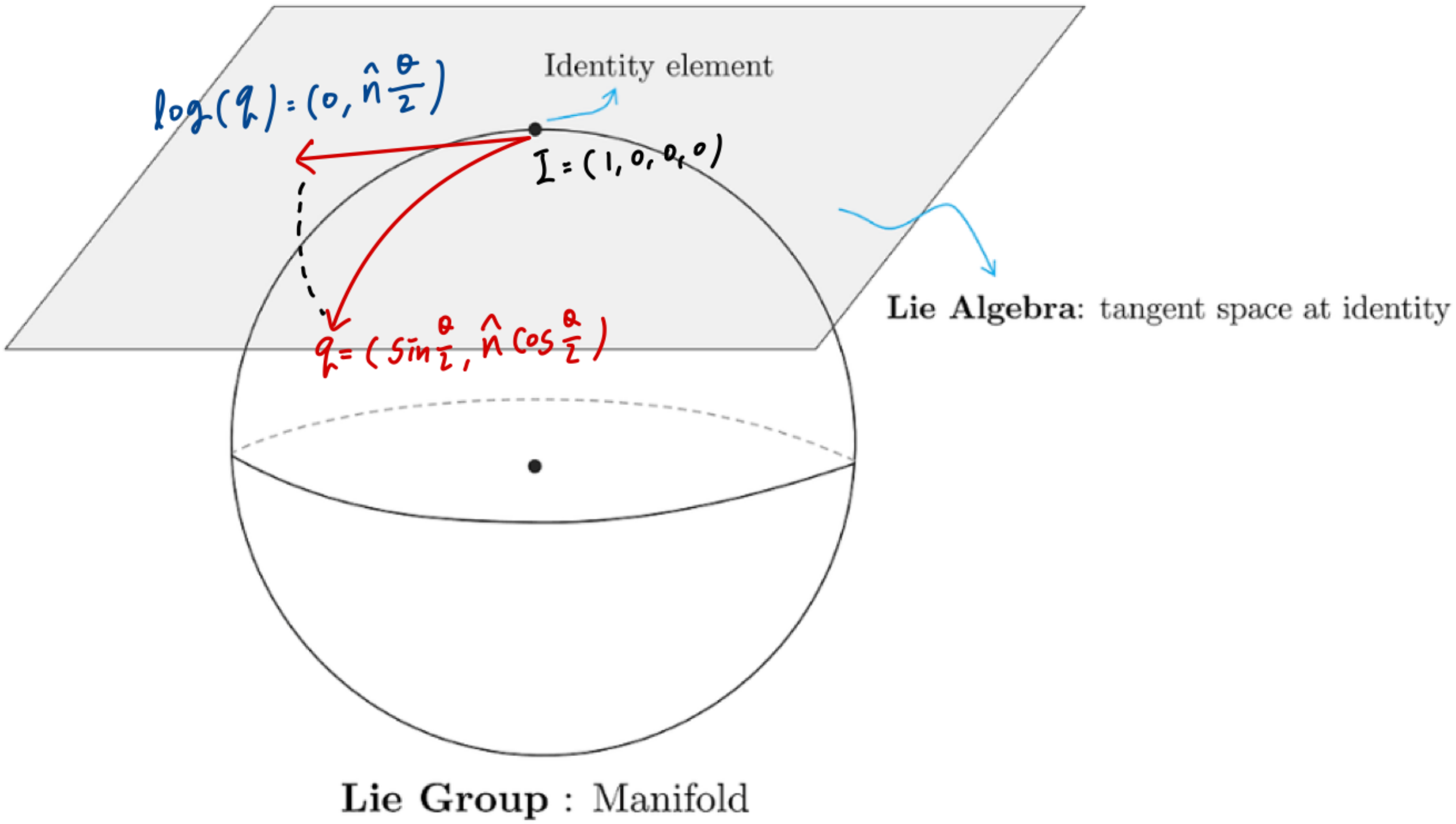
$$log(q) = \hat{n}\frac{\theta}{2}$$

$$exp(\hat{n}\tfrac{\theta}{2}) = q$$

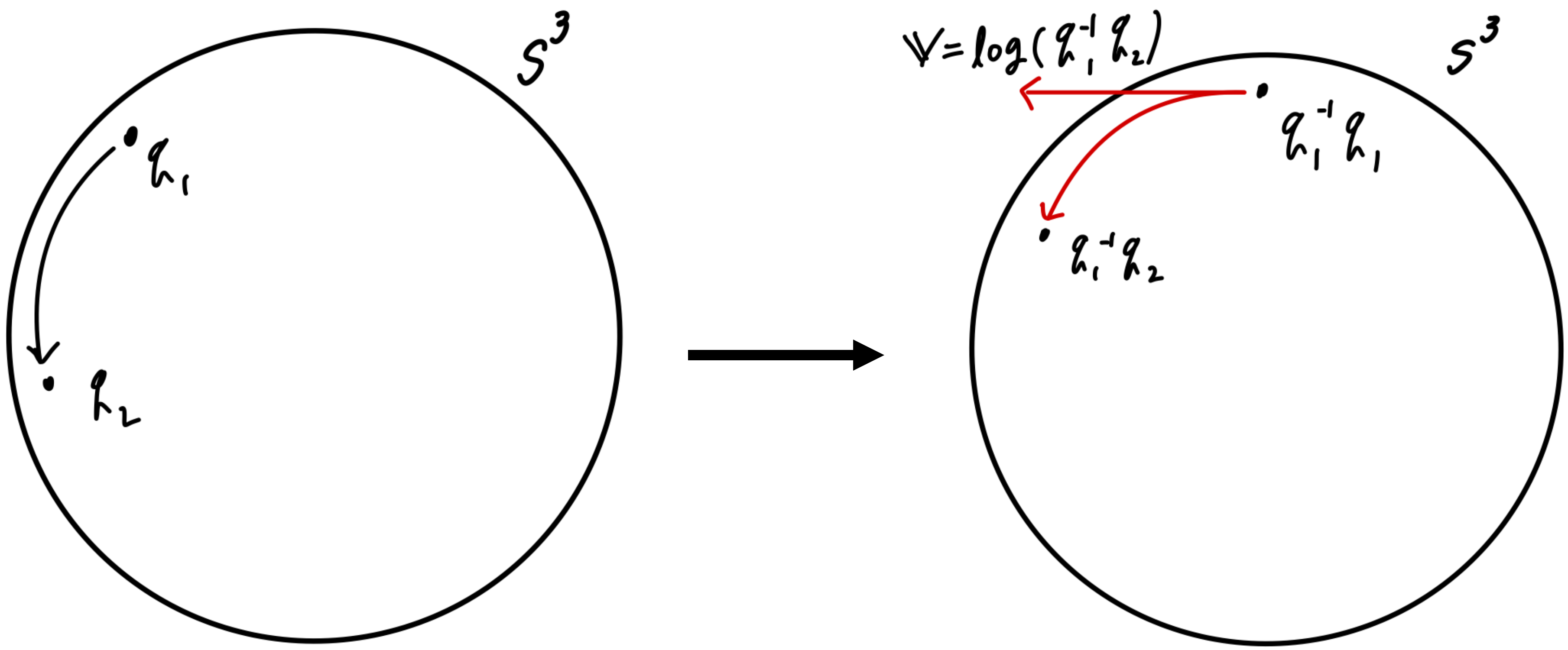




Exponential & Logarithmic Map



How to go from q_1 to q_2



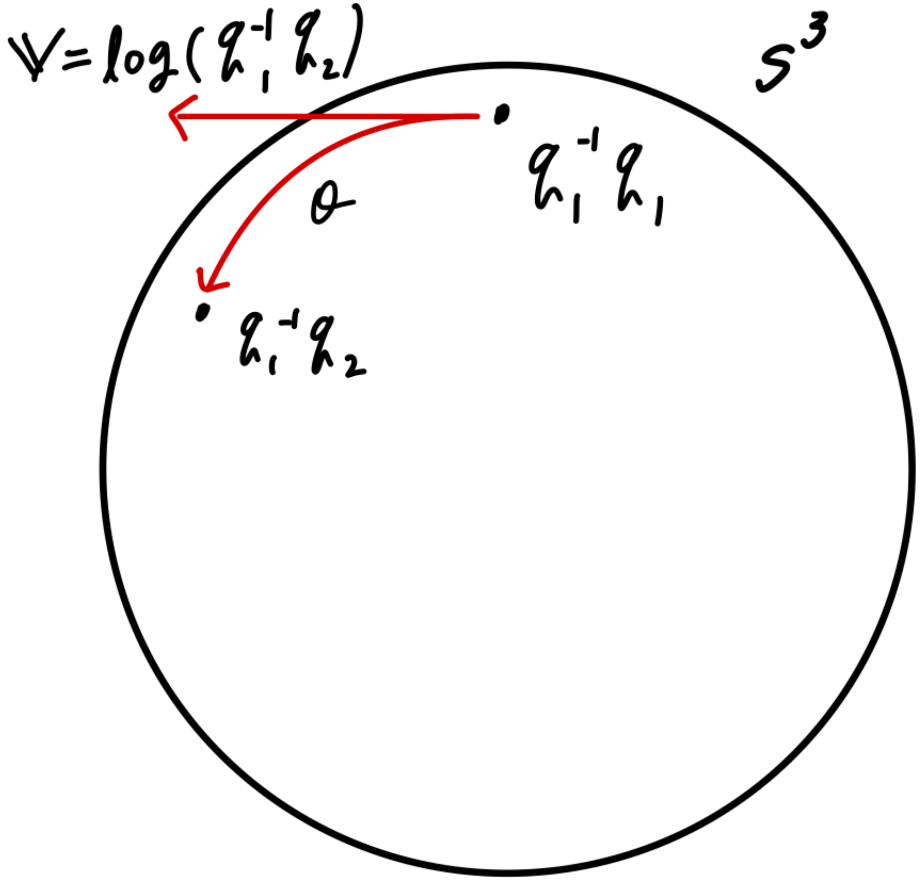
$$q_2 = exp(log(q_1^{-1} q_2))q_1$$

How to go from q_1 to q_2

Rotation vector from q_1 to q_2

$$v = log(q_1^{-1}q_2) \times 2$$

$$q_2 = exp(\frac{v}{2})q_1 = exp(\frac{log(q_1^{-1}q_2)\times 2}{2})q_1$$



Euler angles to Unit quaternion

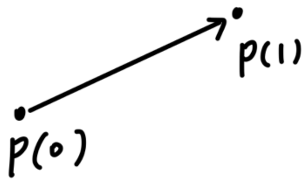
Rotation vector from q_1 to q_2

$$\text{Angles: } a_x, a_y, a_z$$
$$\text{XYZ order: } exp(\frac{a_z z}{2})exp(\frac{a_y y}{2})exp(\frac{a_x x}{2})$$
$$q = q_3 q_2 q_1$$

$$Z_1 Y_2 X_3 = \begin{bmatrix} c_1 c_2 & c_1 s_2 s_3 - c_3 s_1 & s_1 s_3 + c_1 c_3 c_2 \\ c_2 s_1 & c_1 c_3 + s_1 s_2 s_3 & c_3 s_1 s_2 - c_1 s_3 \\ -S_2 & C_2 S_3 & C_2 C_3 \end{bmatrix}$$

Spherical linear interpolation (SLERP)

Interpolation

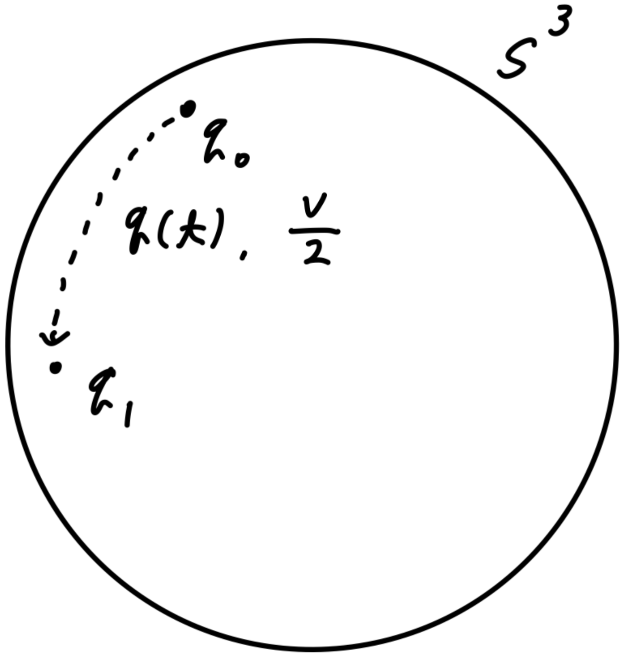


$$p(t) = (1-t)p(0) + p(1)$$

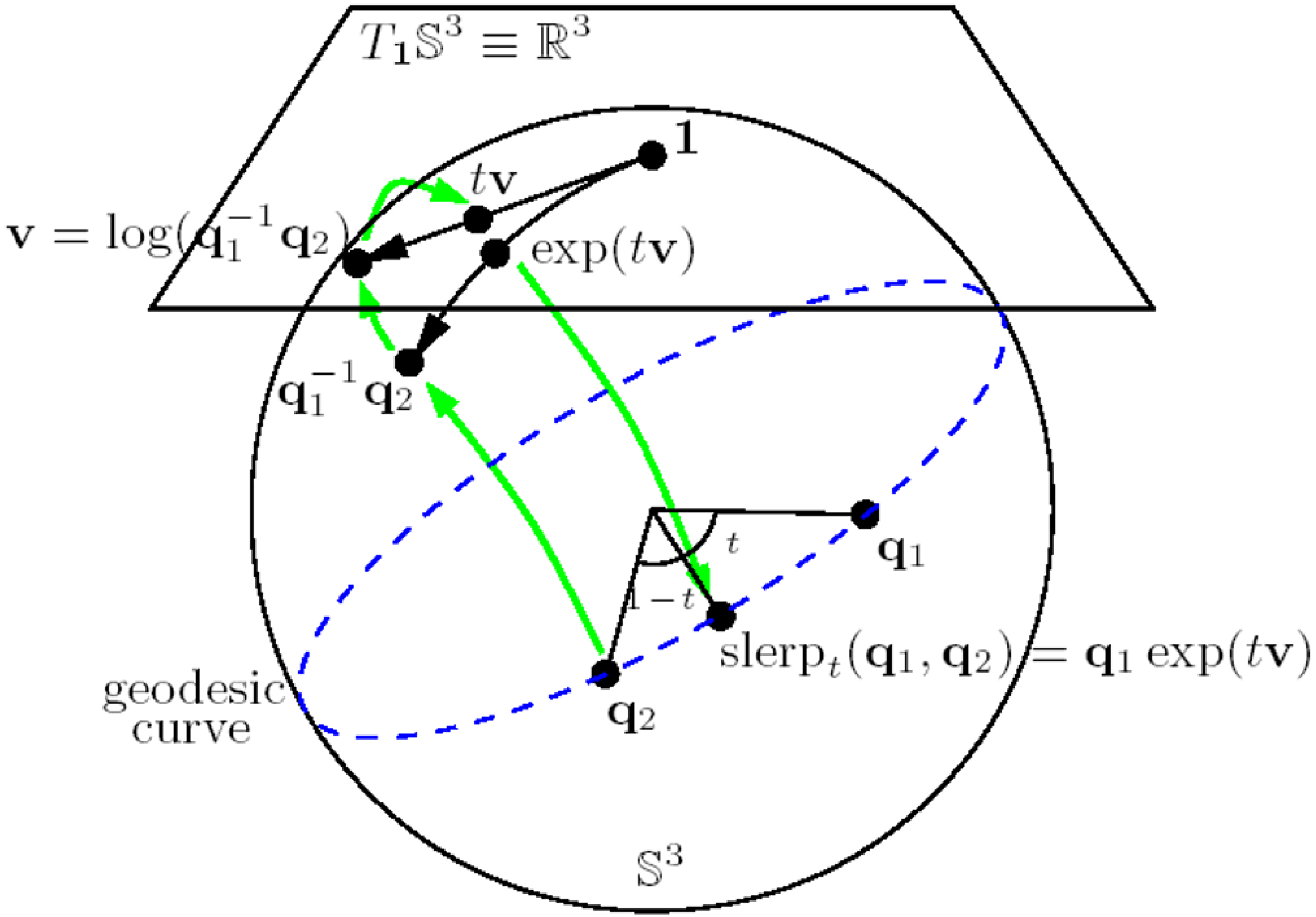
$$= (p(1) - p(0))t + p(0)$$

$$v = log(q_1^{-1}q_2) \times 2$$

$$q(t) = q_1 exp(\frac{tv}{2})$$



Spherical linear interpolation (SLERP)



Q&A